Boundary Layer and a Stabilized Gaseous Discharge in the Presence of Diffuse Radiation

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When bodies with supersonic velocities enter the dense layers of the atmosphere, the radiation of air heated by the shock wave produces an essential influence on the characteristics of the air flow. In Ref. 1 characteristic parameters of this process have been analyzed and a series of examples examined. In this paper we examine the process of interaction of a viscous radiating gas stream with the surface of the body, possessing characteristics of a boundary layer, and possessing the same phenomenon as that exhibited by stabilized gaseous discharges at large pressures. It is assumed that the mean free path of the radiation is small and that the radiation obeys the laws of diffusion.

1. The equations of gasdynamics, with the inclusion of radiation terms, have the following form:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \boldsymbol{v} = 0 \qquad \rho \frac{d\boldsymbol{v}}{dt} = -\operatorname{grad} p + \frac{\partial \sigma_{ik}'}{\partial x_k}$$
$$\rho \frac{d}{dt}\left(i + \frac{v^2}{2}\right) = \operatorname{div}(k^* \operatorname{grad} T) + \sigma_{ik}' \frac{\partial v_i}{\partial x_k} \tag{1}$$
$$p = p(\rho, T)$$

Here i is the enthalpy, σ_{ik}' the viscosity tensor, and k^* the generalized coefficient of heat conduction containing the radiation effect

$$k^* = k_\lambda + k_T$$

while k_T is the ordinary heat conduction coefficient, and $k_\lambda = \frac{3}{4} ac T^3 \bar{l}$ is the coefficient of radiative heat conduction, with c the speed of light, $\bar{l}$ the mean free path of radiation, and a the Stefan-Boltzmann constant.

Relative importance of viscous and heat conduction terms in these equations are given by Reynolds (Re) and Pecley (Pe) numbers. In cases when these two numbers are large, viscosity and heat conduction will essentially be limited only to a small region adjacent to the surface of the body—a boundary layer.

The equations for the boundary layer in the flat stationary case will be:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$
$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{2}$$
$$\rho u \frac{\partial i}{\partial x} + \rho v \frac{\partial i}{\partial y} = u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(k^* \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu u \frac{\partial u}{\partial x}\right)$$
$$\partial p / \partial y = 0 \qquad p = p(\rho T)$$

These equations reflect the fact that the characteristic time for the expansion of the disturbance into the stream on account of viscosity and heat conduction is significantly greater than the time needed for the passage of the gas particles through a characteristic interval along the surface boundary. This treatment also concerns radiative heat transfer, for during this characteristic time, the expansion of the disturbance will be

$$\tau = L^2 \mid k_\lambda / \rho c_p$$

It should be noted that in these equations the coefficients of viscosity and heat conduction appear as certain functions of density and temperature.

System (2) has a number of similarity solutions (see Ref. 3).

For the case of a flat plate (p = const) those solutions have the form

$$\psi = \rho_\infty \sqrt{\nu_\infty u_\infty x} f(\xi) \qquad T = T_\infty t(\xi)$$
$$H = H_\infty h(\xi) \qquad p = p_\infty$$
$$\rho = \rho_\infty l(\xi) \qquad \xi = y\sqrt{u_\infty/\nu_\infty x}$$
$$u = \frac{1}{\rho}\frac{\partial \psi}{\partial y} \qquad v = -\frac{1}{\rho}\frac{\partial \psi}{\partial x} \tag{3}$$
$$H = i + (\partial \psi / \partial y)^2 | 2\rho^2$$
$$\nu = \nu_\infty \bar{\nu}(t,l) \qquad k^* = k_\infty^* \bar{k}(t,l)$$

and the system of equations (2) results in a system of ordinary differential equations

$$f\left(\frac{f'}{l}\right)' = \left[\bar{\mu}\left(\frac{f'}{l}\right)'\right]'$$
$$\frac{fh'}{2} = \frac{k_\infty^* T_\infty}{\rho_\infty \nu_\infty H_\infty}(\bar{k}t')' + \frac{u_\infty}{H_\infty}\left[\frac{\bar{\mu}}{l} f' \left(\frac{f'}{l}\right)'\right]' \tag{4}$$

with the following boundary conditions:

$$\xi = 0 \qquad f = f' = 0 \qquad t = \frac{T_0}{T_\infty}$$
$$\xi \to \infty \qquad f' \to 1 \qquad t \to 1 \qquad l \to 1 \tag{5}$$

if the incident stream has parameters u_∞, p_∞, T_∞, and ρ_∞ and the plate is maintained at a constant temperature T_0. On the other hand the boundary conditions become $\xi = 0$, $f = f' = 0$, $t' = 0$ if the plate is insulated.

2. As was already noted, large Re and Pe numbers are appropriate, since a narrow viscous and thermal boundary layer is formed adjacent to the surface whose thickness is of the order $\delta_\mu \approx L/\sqrt{Re}$ and $\delta_k^* \approx L/\sqrt{Pe}$. In the case when the Re and $Pe = RePr$ numbers are of the same order, that is, when the Prandtl number $Pr \approx 1$, the viscous and thermal effects should be investigated jointly. If, however, $Re \gg Pe \gg 1$ (i.e., when $Pr \ll 1$), the following features are exhibited.[4]

We have a thermal layer of thickness $\sim\delta_k^*$, where the temperature changes into its ultimate order. Inside this layer, in the immediate vicinity of the surface, there is a regular viscous layer, where the thermal source is centered. This viscous layer is so thin that we may assume it to be isothermal; in the thicker thermal layer we may disregard viscosity.

From this point of view, let us examine the possible significance of Prandtl and Peckley numbers in the case when bodies become surrounded by air currents. For the case of radi-

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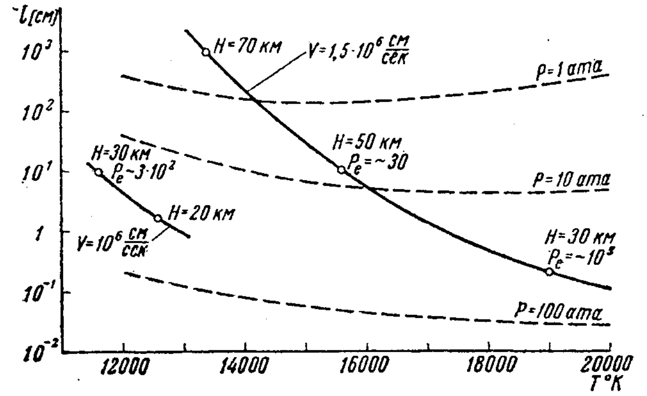

Fig. 1

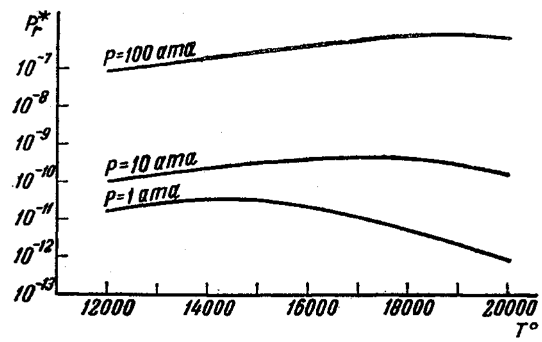

Fig. 2

ative heat transfer, the Prandtl number is given by the formula

$$Pr_\lambda = \mu c_p/k_\lambda = \mu c_p/\tfrac{4}{3}acT^3\bar{l}$$

where the average run of radiation may be determined from the results found in Ref. 1.

If we assume that basically we have a continuous radiation, due to the recombination of ions of oxygen and nitrogen, then with temperatures up to 20,000°K, the average absorption coefficient of the ground state is two orders of magnitude greater than the absorption coefficient of the excited state of atoms, and, therefore, for small thicknesses the frequencies corresponding to the ground state will be "stopped" first. In Figs. 1 and 2, we show the values of $\bar{l}$ and Pr^* represented parametrically for different values of the velocity V of bodies entering the dense layers of the atmosphere for given flight altitudes H. From these diagrams, we see that the curves possess certain regions where $\bar{l}$ is of the order of a few centimeters or less, whereas the Perklay number $Pe^* \gg 1$.

It should be noted that the effect of layer separation will occur due to the presence of a dense stream of high velocity plasma near the surface of the body, and also that this phenomenon will always be conditional on the presence of highly absorbent ingredients in the circumfluent stream, which may arise, for example, due to charring or sublimation of the surfaces.

3. Let us now examine in detail the flow over a flat plate, for the case $Re \gg Pe^* \gg 1$. Equations for the thermal layer will be

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0$$

$$\rho u \frac{\partial i}{\partial x} + \rho v \frac{\partial i}{\partial y} = \frac{\partial}{\partial y}\left(\frac{k^*}{c_p}\frac{\partial i}{\partial y}\right) \tag{7}$$

$$p_\infty = \frac{\gamma - 1}{\gamma}\rho i = \text{const}$$

where the ratio $\gamma = c_p/c_v$ will be assumed constant. The conditions at infinity are $u = u_\infty$, $T = T_\infty$; at the surface we require that $v = 0$ and $T = T_0$. The values of u_0 and ρ_0, together with the result to the foregoing problem, specify the flow in the much thinner viscous layer. Using the transformations $\xi = x$ and $\eta = \int_0^y \frac{\rho}{\rho_0}\,dy$, the first two equations transform into

$$\frac{\partial u}{\partial \xi} + \frac{\partial \bar{v}}{\partial \eta} = 0 \qquad u\frac{\partial u}{\partial \xi} + \bar{v}\frac{\partial u}{\partial \eta} = 0 \tag{8}$$

where $\bar{v} = u(\partial\eta/\partial x) + (v\rho/\rho_0)$ with the conditions $u|_{\eta\to 0} = u_\infty$, $\bar{v}|_{\eta=\infty} = 0$, from which one concludes that $u = u_\infty$ and $\bar{v} = 0$.

The energy equation is then

$$\frac{\partial i}{\partial \xi} = \frac{1}{\rho_0{}^2 u_\infty}\frac{\partial}{\partial \eta}\left(\frac{\rho k^*}{c_p}\frac{\partial i}{\partial \eta}\right) \tag{9}$$

The coefficient k^*, appearing as a function of enthalpy, may be represented approximately on disjoint intervals of the independent variable by ci^n. Then, on introducing the new variable

$$\tau = \frac{\eta}{\sqrt{2\dfrac{\gamma}{\gamma - 1}\dfrac{c}{c_p}\dfrac{p}{\rho_0{}^2 u_\infty} i_0{}^{n-1}\xi}}$$

we get from (9) an ordinary nonlinear differential equation in the dimensionless form

$$-\tau i' = (i^{n-1}i')' \tag{10}$$

Study of this equation shows that when $\tau \to 0$ and $\tau \to \infty$, i tends toward a constant value, and its equation has the invariant form

$$i = \alpha^{2/n}y(\alpha\varkappa) \tag{11}$$

This fact may be used when one carries out a numerical integration.

Let us now examine the position of the following problems.

a. A gas on the surface of the plate has a given enthalpy i_0, and the profile of i changes into i_∞ when $\tau \to \infty$, while at the same time the region problem may be solved numerically under the conditions $\tau = 0$, $i = i_0$, and $\partial i/\partial\tau = \beta$, where β is chosen such that as $\tau \to \infty$ the function i goes to i_∞.

b. The mean free path of radiation usually increases sharply with decreasing temperatures. Therefore, it is expedient to pose the following problem with stated conditions when $\tau = 0$ and $\tau \to \infty$: at some τ^* i assumes the value i^*, while the radiation loses the features of thermal radiative conductivity, and the amount of radiation for this

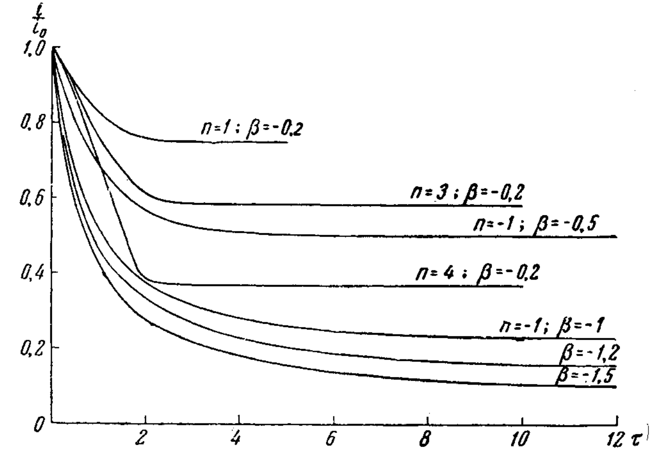

Fig. 3

energy level ($F = \sigma T^{*4}$) is small; then provided $\tau = \tau^*$ we will have

$$k^* \frac{\partial i}{\partial \tau}\bigg|_{\tau=\tau^*-0} \approx k_T \frac{\partial i}{\partial \tau}\bigg|_{\tau=\tau^*+0} \qquad (12)$$

where k_T is the ordinary heat conduction coefficient. Obviously, these stipulations may be posed also with small values of τ^*, when surface cooling occurs.

The viscous layer generates heat which is transferred to the wall and the thermal layer according to the equation

$$q = -\frac{1}{2}\frac{\partial}{\partial x}\int_0^\infty \rho_0 u(u^2 - u_\infty{}^2)dy \qquad (13)$$

and which may be found by calculations for an incompressible viscous layer.[2] The thermal stream toward the wall may then be determined from the formula

$$q_0 = q - q_T = q - \frac{k^*}{c_p}\frac{\partial i}{\partial y}\bigg|_{y=0} \qquad (14)$$

From these conditions, we may find the parameters of stream flow for different allowable values of q_0 and i_0. In particular, when $q_0 = 0$ we have a thermally insulated plate.

Figure 3 gives curves, which were calculated with the help of high-speed digital computer for case a for various values of β and n.

4. We conclude with a discussion of a process of the same nature which can occur during longitudinal stabilization of a gas discharge in the presence of a large pressure when the magnetic Reynold's number is large.[3]

In this case, the magnetic Reynold's number has a role which is precisely analogous to that of an ordinary Reynold's number in the case analyzed in the foregoing: compressed by a large convective stream which is a good internal conductor of electric current, heat sources will be concentrated near the discharge axes in a narrow region, having a thickness of the order of $\delta \approx L/\sqrt{Re\ m}$; further away from the axes will extend a wider region of the thermal layer, containing no currents and described by Eq. (7). Heat generated by the electric current layer $q = 4\pi\int_0^\infty \nu_m j^2 dy$ (in the flat case) will be transferred to the thermal layer. In this expression j is the electrical current density and V the magnetic viscosity.

References

[1] Zhigulev, V. N., Romishevskii, Ye. A., and Vertushkin, V. K., "Role of radiation in modern gasdynamics," Inzh. Zh. (Eng. J.) **1**, 60–83 (1961); see AIAA J. **1**, No. 6, 1473–1485 (1963).

[2] Konchin, N. E., Kibel', I. A., and Roze, N. V., *Teoreticheskaya Gidromekhanika* (*Theoretical Hydromechanics*) (Gostekhizdat, State Technical Literature Press, Moscow, 1948).

[3] Zhigulev, V. N., "Magnetic boundary layer in a compressible fluid," Izv. Acad. Nauk SSSR, Otd. Tekh. Nauk (Bull. Acad. Sci. USSR, Div. Tech. Sci.), No. 5 (1960).

[4] Kogan, M. N., "Flows with high heat conductivity," Dokl. Acad. Nauk SSSR (Rept. Acad. Sci. USSR) **128**, No. 3 (1959).

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Integral Kinetic Equations of the Theory of Monatomic Gases in the Presence of an External Field of the Forces of Mass

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INTEGRAL kinetic equations of the theory of monatomic rarefied gases were derived in Refs. 1 and 2. Problems in the aerodynamics of monatomic rarefied gases were also stated in Ref. 2.

The integral kinetic equations given in Refs. 1 and 2 were of the following form:

$$f(\bar r,\bar u,t) = (1/|u_n|)\,\tilde\Phi(\bar r_s,\bar u,t)\Pi(\bar r,\bar u,t,\tau_s) + \int_{\tau_s}^{t}\Phi(\bar r - \bar u(t-\tau),\bar u,\tau)\Pi(\bar r,\bar u,t,\tau)d\tau \qquad (1)$$

$$\Pi(\bar r,\bar u,t,\tau) = e^{-\int_\tau^t\left[\iiint_{-\infty}^{+\infty}|\bar u - \bar u_1|\sigma(|\bar u - \bar u_1|)f(\bar r - \bar u(t-q),\bar u_1,q)d\omega_1\right]dq} \qquad (2)$$

$$\Phi(\bar r,\bar u,t) = \frac{1}{2}\iiint\!\!\iiint_{-\infty}^{+\infty}|\bar u_1 - \bar u_2|\,\sigma(|\bar u_1 - \bar u_2|)f(\bar r,\bar u_1,t)\,\times f(\bar r,\bar u_2,t)T(\bar u_1,\bar u_2,\bar u)d\omega_1 d\omega_2 \qquad (3)$$

$$\tilde\Phi(\bar r_s,\bar u,t) = \iiint_{(u_1)_n<0} f(\bar r_s,\bar u_1,t)\,|(u_1)_n|\tilde T\,(\bar u_1,\bar n,\bar u,\theta)d\omega_1 \qquad (4)$$

In these equations f is the distribution function; Π the probability of free motion; Φ the internal generation function; $\tilde\Phi$ the boundary generation function; T the internal shock transformant; $\tilde T$ the boundary shock transformant; σ the collision cross section, dependent on the relative velocity of colliding particles; $\bar u_1$ and $\bar u_2$ the integration variables (vector) in velocity space; $d\omega_1$ and $d\omega_2$ are the volume elements in velocity spaces $\bar u_1$ and $\bar u_2$; τ and q are scalar parameters; $\bar n$ is the external normal to the surface of the streamlined body in the point with radius vector $\bar r_s$ at the moment of time τ_s; θ the surface temperature in the point with radius vector $\bar r_s$ at the moment of time τ_s; and $\bar r_s$ and τ_s are certain functions of $\bar r$ and t. The remaining notations and definitions are assumed known from Refs. 1 and 2.

It was assumed in Refs. 1 and 2 that the external field of the forces of mass does not affect the moving gas. The present paper determines a system of equations which is a generalization of Eqs. (1–4) and is applicable to the case of motion of the gas when it is affected by a constant external field of the force of mass.

However, due to the exponential decrease of the probability of free motion Π, the equations derived in the present paper are valid also in variable fields of force if the variation of the fields is negligibly small at distances of the order of magnitude of 5–10 mean free paths during periods of time equal to 5–10 average time intervals between the collision of atoms.

The derivation of Eqs. (1–4) implies that the validity of (3) and (4) is not dependent on the assumed absence of the

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